Science
SM358 The Quantum World

SM358

Tutor-Marked Assignment 02

Contents **Cut-off date**

Please send all your answers to the tutor-marked assignment (TMA) to reach your tutor by 12 noon (UK local time) on or before the cut-off date shown on the SM358 website. Your TMAs should be submitted through the eTMA system unless there are difficulties which prevent you from doing so. In these circumstances, you must negotiate with your tutor to get their agreement to submit your assignment on paper. The eTMA system allows for eTMA submission directly to the university 24 hours a day, and either gives you confirmation that your eTMA has been submitted successfully or, if there has been a problem, an error message informing you of the problem and what steps you can take to overcome it. If you submit online you must keep your receipt code in case you need to prove successful submission.

General information about policy and procedure is in the *Assessment Handbook* which you can access from your StudentHome page. However, there are a number of ways in which SM358 eTMA submission differs from what is described there. These are described in the document *Producing eTMAs for Level 3 physics and astronomy modules* on the SM358 website. See also the SM358 *Introduction and Guide* for module-specific information.

Of particular importance is the test submission, TMA 00. This will enable you to familiarize yourself with the system and allow your tutor to check that the format in which you save your TMAs is compatible with their own computer software. It is your responsibility to make sure that you submit documents in a compatible format and we strongly recommend that you submit TMA 00. TMAs submitted in an incorrect format may not be marked.

If you are submitting a paper copy, please allow sufficient time in the post for the assignment to reach its destination on or before the cut-off date. We strongly advise you to use first-class post and to ask for proof of postage. Do not use recorded delivery or registered post as your tutor may not be in to receive it. Keep a copy of the assignment in case it goes astray in the post. You should also include an appropriately completed assignment form (PT3). You will find instructions on how to fill in the PT3 form in the *Assessment Handbook*. Remember to fill in the correct assignment number (02).

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5.1 PT/TeX

This booklet provides advice about submission of TMA answers as well as the questions for TMA 02. Although the marks for your assignments do not count directly towards your SM358 result, they are an essential part of your learning and you are required to engage satisfactorily with them. Please refer to the SM358 *Introduction and Guide*, for additional information about the module assessment.

Assignment cut-off dates

The cut-off dates for the assignments provide an important element of pacing for your study of SM358 and they are spread fairly uniformly through the year, leading up to the exam. **You should regard these dates as fixed points**. *Any extension to a TMA cut-off date requires prior permission from your tutor, which may not always be given. Extensions may be granted in exceptional circumstances but it will never be possible to have an extension of more than 3 weeks.* Your tutor will, of course, be willing to discuss with you the best strategies for catching up if you have fallen behind, and should be able to help with questions if you are stuck.

Plagiarism

You are encouraged to discuss the module materials and assignment questions with other students, but the answers to the assignment questions must be your own work. This does not preclude you from making judicious use of material from other sources, but you must acknowledge such use by giving the author's name, the year of publication, the name of the publication in which it appears (or the website address), and the edition or volume number and the page number. However, there is no need to give references for standard equations in the SM358 texts. You are advised to read the University's guidelines on plagiarism, contained in the *Assessment Handbook*, available online from StudentHome.

To check that all students are working in a fair and academically appropriate manner, the Open University is currently using some text-comparison software to detect potential cases of plagiarism in work that is submitted for assessment. Details of how this is implemented in this module are given on the SM358 website.

Further general advice on answering SM358 assignment questions is given in the first assignment booklet.

TMA 02 CUT-OFF DATE: SEE SM358 STUDY PLANNER

This assignment is related to Chapters 5–7 of Book 1 and Chapter 1 of Book 2. Your answers for this assignment will provide evidence of your achievement of many of the learning outcomes, as listed in the *Introduction and Guide*, Section 6.5. In particular, this assignment tests *Knowledge and understanding* outcomes 1–4, *Cognitive skills* outcomes 1–3 and all three of the *Key skills* outcomes.

Question 1

This question carries 33% of the marks for this assignment. It relates mainly to Chapter 5 of Book 1, and particularly to Achievement 5.7, and to Chapter 4 of Book 1, particularly Achievement 4.6.

This question refers to a harmonic oscillator with length parameter a. In terms of raising and lowering operators, the operator $\widehat{\mathrm{p}}_x^4$ for such an oscillator can be expressed as

$$\widehat{\mathrm{p}}_x^4 = \frac{\hbar^4}{4a^4}\left(\widehat{\mathrm{A}} - \widehat{\mathrm{A}}^\dagger\right)^4.$$

The right-hand side of this equation can be expanded to give a sum of terms, each of which is a product of four raising or lowering operators. However, most of these terms have zero expectation value in the ground state of the oscillator.

(a) Explain why any term (such as $\widehat{\mathrm{A}}\widehat{\mathrm{A}}^\dagger\widehat{\mathrm{A}}^\dagger\widehat{\mathrm{A}}$) with a lowering operator on the extreme right has zero expectation value in the ground state of a harmonic oscillator. (5 *marks*)

(b) Explain why any term (such as $\widehat{\mathrm{A}}\widehat{\mathrm{A}}^\dagger\widehat{\mathrm{A}}^\dagger\widehat{\mathrm{A}}^\dagger$) with *unequal* numbers of raising and lowering operators has zero expectation value in the ground state of a harmonic oscillator.

(8 *marks*)

(c) Taking into account the results of parts (a) and (b), the expectation value of p_x^4 in the ground state of a harmonic oscillator can be expressed as

$$\langle p_x^4\rangle = \frac{\hbar^4}{4a^4}\left[\int_{-\infty}^{\infty}\psi_0^*(x)\left(\widehat{\mathrm{A}}\widehat{\mathrm{A}}\widehat{\mathrm{A}}^\dagger\widehat{\mathrm{A}}^\dagger + \widehat{\mathrm{A}}\widehat{\mathrm{A}}^\dagger\widehat{\mathrm{A}}\widehat{\mathrm{A}}^\dagger + \widehat{\mathrm{A}}^\dagger\widehat{\mathrm{A}}\widehat{\mathrm{A}}\widehat{\mathrm{A}}^\dagger\right)\psi_0(x)\,\mathrm{d}x\right],$$

where $\psi_0(x)$ is the ground-state energy eigenfunction of the oscillator. Starting from this formula show that, in the ground state,

$$\langle p_x^4\rangle = \frac{3\hbar^4}{4a^4}.$$

(10 *marks*)

(d) Chapter 5 shows that, in the ground state of the oscillator,

$$\langle p_x^2\rangle = \frac{\hbar^2}{2a^2}.$$

Use this fact, together with the answer to part (c), to find the uncertainty of the kinetic energy in the ground state of a harmonic oscillator. Express your answer as a multiple of the ground-state energy $E_0 = \frac{1}{2}\hbar\omega_0$, where ω_0 is the classical angular frequency.

(10 *marks*)

Question 2

This question carries 22% of the marks for this assignment and relates to Chapter 6 of Book 1, particularly Achievements 6.2, 6.3 and 6.5.

A particle is in the potential well of a harmonic oscillator in a state described by the initial wave function

$$\Psi(x, 0) = \frac{1}{\sqrt{2}}\big(\psi_1(x) - \psi_3(x)\big),$$

where $\psi_1(x)$ and $\psi_3(x)$ are real normalized energy eigenfunctions with quantum numbers $n = 1$ and $n = 3$ respectively.

(a) Assuming that the system remains undisturbed, obtain an expression for $\Psi(x, t)$ that is valid for all $t \geq 0$. Express your answer in terms of the functions $\psi_1(x)$, $\psi_3(x)$ and ω_0, the classical angular frequency of the oscillator. (5 *marks*)

(b) Remembering that $\psi_1(x)$ and $\psi_3(x)$ are real, write down the corresponding expression for $\Psi^*(x, t)$ and hence find the probability density function for the particle at any time $t \geq 0$. Express your answer in terms of $\psi_1(x)$, $\psi_3(x)$, t and ω_0. (7 *marks*)

(c) Given that $\psi_1(x)$ and $\psi_3(x)$ are both odd functions, use your answer to part (b) to show that $\langle x \rangle = 0$ at all times. (7 *marks*)

(d) In spite of your answer to part (c), the probability density in part (b) does vary cyclically in time (it breathes in and out, rather than shuttling to-and-fro across the well). Express the period of this cyclic motion in terms of the classical angular frequency, ω_0. (3 *marks*)

Question 3

This question carries 22% of the marks for this assignment and relates to Chapter 7 of Book 1, and particularly to Achievements 7.3 and 7.4.

A steady beam of particles travels in the x-direction and is incident on a finite square barrier of height V_0, extending from $x = 0$ to $x = L$. Each particle in the beam has mass m and total energy $E_0 = 2V_0$. Outside the region of the barrier, the potential energy is equal to 0.

In the stationary-state approach, the beam of particles is represented by an energy eigenfunction of the form

$$\psi(x) = \begin{cases} A\mathrm{e}^{\mathrm{i}k_1x} + B\mathrm{e}^{-\mathrm{i}k_1x} & \text{for } x < 0 \\ C\mathrm{e}^{\mathrm{i}k_2x} + D\mathrm{e}^{-\mathrm{i}k_2x} & \text{for } 0 \leq x \leq L \\ F\mathrm{e}^{\mathrm{i}k_1x} \quad \text{for } x > L \end{cases}$$

where A, B, C, D and F are constants and k_1 and k_2 are wave numbers appropriate for the different regions.

(a) Express the wave numbers k_1 and k_2 in terms of V_0, m and $\hbar$. (6 *marks*)

(b) Find four equations that specify the continuity boundary conditions for $\psi(x)$ and $\mathrm{d}\psi/\mathrm{d}x$ at $x = 0$ and $x = L$. (8 *marks*)

(c) Now consider the special case where $k_2L = \pi/2$ (which does *not* correspond to a transmission resonance). Use your answer to part (b) to show that

$$\frac{A + B}{A - B} = \frac{k_1}{k_2}\frac{C + D}{C - D} = \frac{k_1^2}{k_2^2}.$$

Use your answer for part (a) and the above equation to find B in terms of A. Hence calculate the reflection coefficient, R, of the beam and deduce the value of the transmission coefficient, T. (8 *marks*)

Question 4

This question carries 23% of the marks for this assignment and relates to Chapter 1 of Book 2, particularly Achievements 1.3 and 1.4.

(a) If $\widehat{\mathrm{B}}$ is a Hermitian operator and $|\Psi\rangle$ represents the bound state of a particle in a one-dimensional potential energy well, show that

$$\langle B^2\rangle = \langle \widehat{\mathrm{B}}\Psi|\widehat{\mathrm{B}}\Psi\rangle,$$

including each step in your reasoning. Use this result to show that the expectation value of the kinetic energy of the particle in the state $|\Psi\rangle$ is

$$\langle E_{\text{kin}}\rangle = \frac{1}{2m}\langle \widehat{\mathrm{p}}_x\Psi|\widehat{\mathrm{p}}_x\Psi\rangle,$$

where $\widehat{\mathrm{p}}_x$ is the momentum operator and m is the mass of the particle. (5 *marks*)

(b) By interpreting $\langle \widehat{\mathrm{p}}_x\Psi|\widehat{\mathrm{p}}_x\Psi\rangle$ in terms of an integral over x, express $\langle E_{\text{kin}}\rangle$ in terms of an integral involving $|\partial\Psi/\partial x|$. Confirm explicitly that your answer cannot be negative in value. (7 *marks*)

(c) Suppose that $\widehat{\mathrm{p}}_x|\Psi\rangle \equiv |\widehat{\mathrm{p}}_x\Psi\rangle$ is expressed in terms of the complete set of orthonormal vectors $|\phi_n\rangle$:

$$|\widehat{\mathrm{p}}_x\Psi\rangle = \sum_n c_n|\phi_n\rangle,$$

where the c_n are complex coefficients (which are not subject to any normalization constraint). Show that

$$\langle E_{\text{kin}}\rangle = \frac{1}{2m}\sum_n |c_n|^2,$$

being careful to show all the steps in your reasoning, with appropriate use of the Kronecker delta symbol. (8 *marks*)

(d) Find an expression for $\langle E_{\text{kin}}\rangle$ in the special case where

$$|\widehat{\mathrm{p}}_x\Psi\rangle = \frac{\hbar}{L}\left(3|\phi_1\rangle - 2\,\mathrm{i}|\phi_2\rangle\right),$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal vectors and L is a constant length. (3 *marks*)

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